Utility Theory:
Foundation and Usage in Decision Making

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Outline

1. Judgement and Decision Making
    - Approaches
2. Utilitarianism
    - Foundation of UT
3. Utility Theory
    - What is Utility ? Why Utility ? How to measure it ?
    - Total and Marginal Utility
    - Utility-Maximizing Rule
    - Utility Axiomatization in Game Theory
    - Risk Aversion and Uncertainty
    - Utility Models and Representation
    - Preferences Elicitation Approaches
    - Example of Preferences Elicitation
    - Limits of Utility Theory

Judgement and Decision making

- The goal is to make a selection among a number of choices or to evaluate opportunities
- Existing approaches:
    1. Classical Decision Theory
    2. Bounded Rationality
    3. Elimination by Aspect
    4. Biases and heuristics
- The theoretical foundation :
    - Economic theory, supply and demand
    - Theory of Utilitarianism
    - Decision Making
    - Game theory

Judgement and Decision making

- The goal is to make a selection among a number of choices or to evaluate opportunities
- Existing approaches:

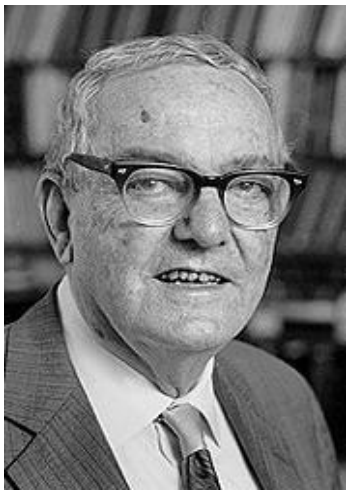
1. Classical Decision Theory
    - Based on the assumption of rationality: People make their choices to maximize some value
    - The goal of human action is to seek pleasure and avoid pain
    - Based on the individual's judgement rather than on objective criteria (subjective utility)
    - Based on the individual's estimates of likelihood, rather than on objective statistical computations (subjective probability)
    - Established as the mathematical models of decision making:

        (-) Does not take into consideration the psychological aspects

Judgement and Decision making

- The goal is to make a selection among a number of choices or to evaluate opportunities
- Existing approaches:

2. Bounded Rationality (Herbert Simon)
    - Humans are boundlessly rational in decision making and limited by their cognitive capabilities
    - Satisfying: Consider the options one by one, and then select the option that is satisfactory or just good enough to meet our minimum level of acceptability



Judgement and Decision making

- The goal is to make a selection among a number of choices or to evaluate opportunities
- Existing approaches:

3. Elimination by aspect
    - Focus on one aspect (attribute) of the available options, and form a minimum criterion for that aspect
    - Eliminate all the options that do not meet that criterion

Judgement and Decision making

- The goal is to make a selection among a number of choices or to evaluate opportunities
- Existing approaches:

4. Biases and Heuristics (Tversky and Kahneman, Gigerenzer)
    - It is most likely that people make decisions based on biases and heuristics (shortcuts)
    - These mental shortcuts lighten the cognitive load of making decisions, although they allow a much greater chance of error



Judgement and Decision making

- In the following, we focus on the classical approach through its economical and mathematical foundations
  - Utilitarianism
  - Utility Theory
    - What is Utility ? Why Utility ? How to measure it ?
    - Total and Marginal Utility
    - Utility-Maximizing Rule
    - Utility Axiomatization in Game Theory
    - Risk Aversion and Uncertainty
    - Utility Models and Representation
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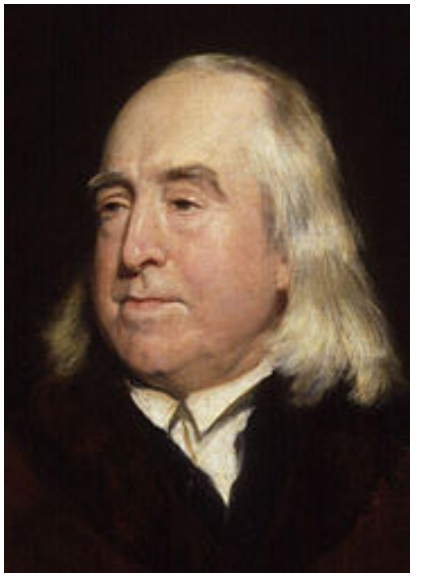
Utilitarianism



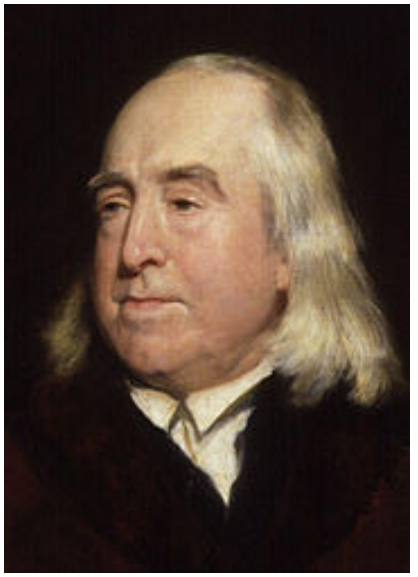
- Jeremy Bentham: the father of utilitarianism

- His central Philosophy:
  - Individual wants and interests *must* be identified with the general *interest* of the *society* as a whole.
  - Bentham's principle of *utilitarianism* asserts that "human conduct should be directed toward maximizing the happiness (surplus of pleasures over pain) of the greatest number of people."

    A History of Economic Theory and Method (p. 125)
  - "An action then may be said to be conformable to the principle of utility (meaning with respect to the community at large) when the tendency it has to augment the happiness of the community is greater than any which it has to diminish it."

    Principles of Morals and Legislation (p. 18)

Utilitarianism



- Jeremy Bentham: the father of utilitarianism

- His central Philosophy:
  - Through the mean of “Utility”, it is possible to discover how prices result from interactions between buyers and sellers
  - Law of supply and demand

Utility Theory

- Situations subject to “preferences”
  - Preferences ? Choices ? Ordering ?
- How to determine the benefits or satisfaction a person receives consuming a good or service ?
  - ➔ Consider the benefit or satisfaction from consuming a good or a service as a “utility”
- What is a utility ?
  - Describes the desirability of preference that individuals or societies have for a given outcome
  - It is a quantitative measure of the attractiveness of a potential outcome

Utility Theory

- Why Utility ?
  - Example: Gamblers cannot use the same rule as to evaluate the gamble, its outcomes, etc.
  - The determination of the value of an item must not be based on its price, but rather on the particular circumstances of the person making the estimate (subjectivity)
- How to measure it ?
  - Cardinal: Measuring utility in “utils”

    Example: Jack derives 10 “utils” from having one slice of pizza but only 5 “utils” from having a burger
  - Ordinal: Measuring utility by comparison

    Example: Jill prefers a burger to a slice of pizza and a slice of pizza to a hotdog

Utility Theory

- Total vs. Marginal Utility
- Total Utility: is the total benefit a person gets from the consumption of goods
- The total utility from a good increases as the quantity of the good increases (increase in the consumption)
  - Example: as the number of movies seen in a month increases, the total utility from movies increases

Lisa's Utility from Movies and Soda

Movies			Soda		
Quantity (per month)	Total utility	Marginal utility	Cases (per month)	Total utility	Marginal utility
0	0	50	0	0	75
1	50	40	1	75	48
2	90	32	2	123	36
3	122	28	3	159	24
4	150	26	4	183	22
5	176	24	5	205	20
6	200	22	6	225	13
7	222	20	7	238	10
8	242	17	8	248	7
9	259	16	9	255	5
10	275		10	260	

Utility Theory

- Total vs. Marginal Utility
- Marginal Utility: is the change in total utility that results from a one-unit increase in the quantity of a good consumed
- As the quantity consumed of a good increases, the marginal utility from consuming it decreases
- We call this decrease in marginal utility as the quantity of the good consumed increases the principle of diminishing marginal utility
  - Example: as the number of movies seen in a month increases, the marginal utility from movies decreases
  - (from the next incremental unit)

Lisa's Utility from Movies and Soda

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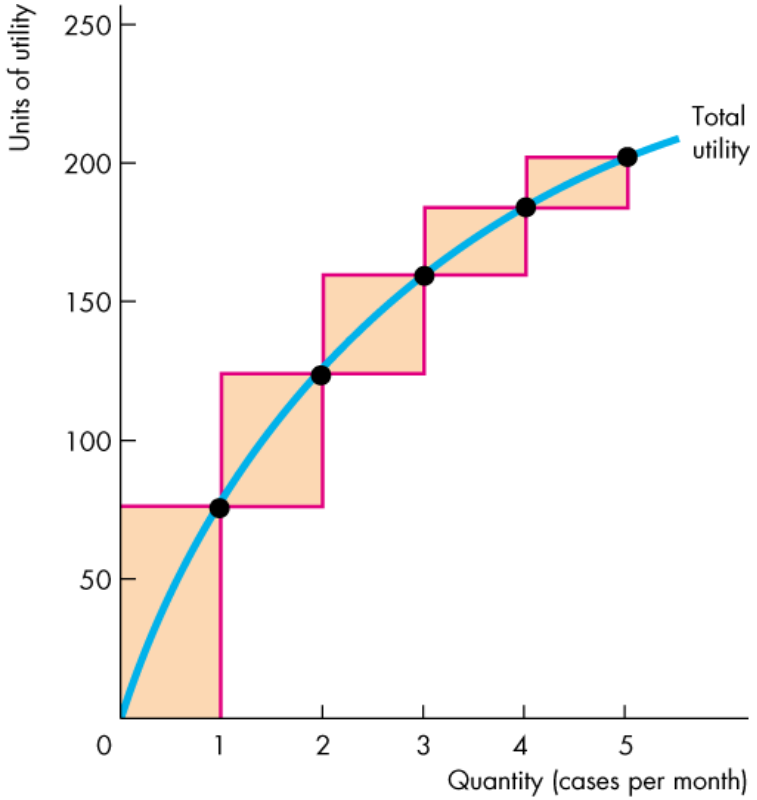
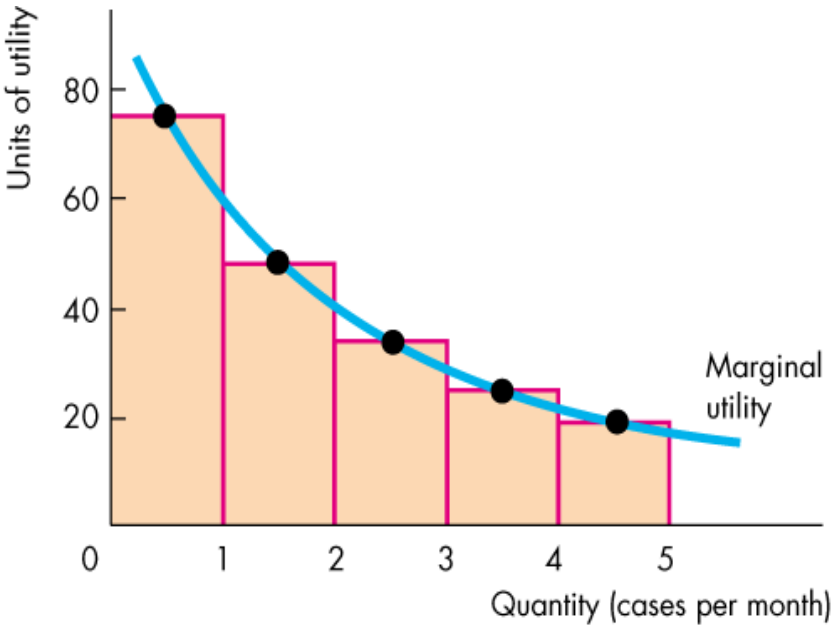
Utility Theory

- Total vs. Marginal Utility
  - Marginal utility is the utility that a consumer derives from the last unit of a consumer good he/she consumes (during a given consumption period), ceteris paribus.
  - Total utility is the total utility a consumer derives from the consumption of all of the units of a good or a combination of goods over a given consumption period, ceteris paribus
  - ➔ Total utility = Sum of marginal utilities

Utility Theory

- Total vs. Marginal Utility



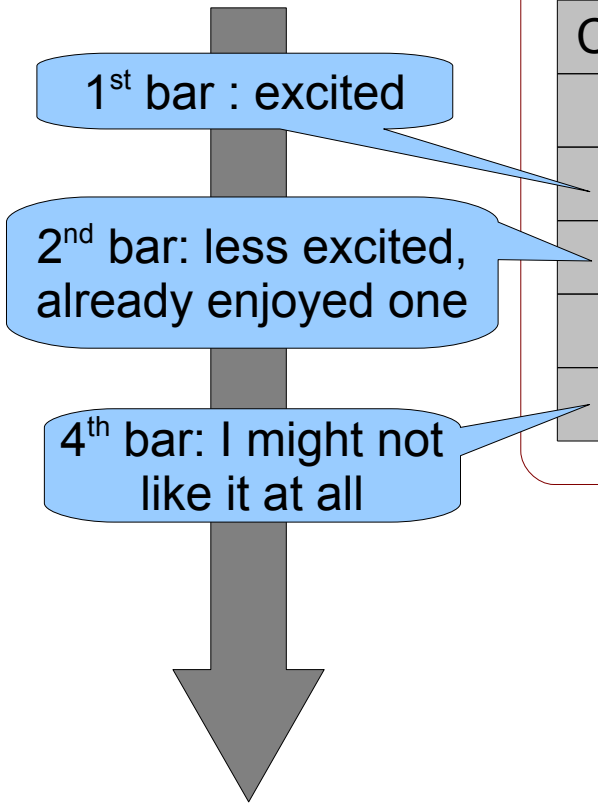


Total utility increases with the **increase** of the soda consumption

As the quantity of soda increases, the marginal utility from soda **diminishes**

Utility Theory

- Utility-Maximizing Rule



Chocolate Bars	
Q	MU
1	100
2	80
3	60
4	40

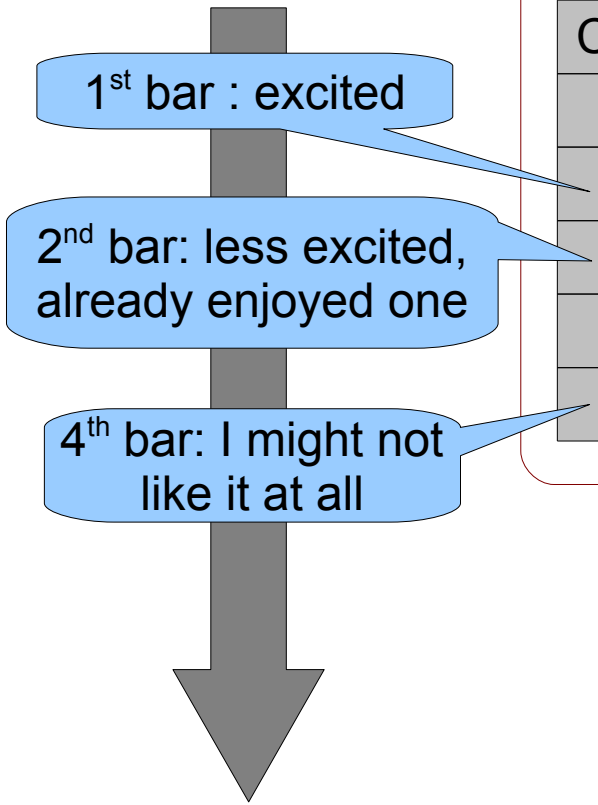
Fruits (1lb)	
Q	MU
1	120
2	100
3	50
4	20

Q: How do we allocate our money given the marginal utilities ?
Fruits over chocolate bars ?

Diminishing utility (benefit) as we are getting more incremental units

Utility Theory

- Utility-Maximizing Rule



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4	20

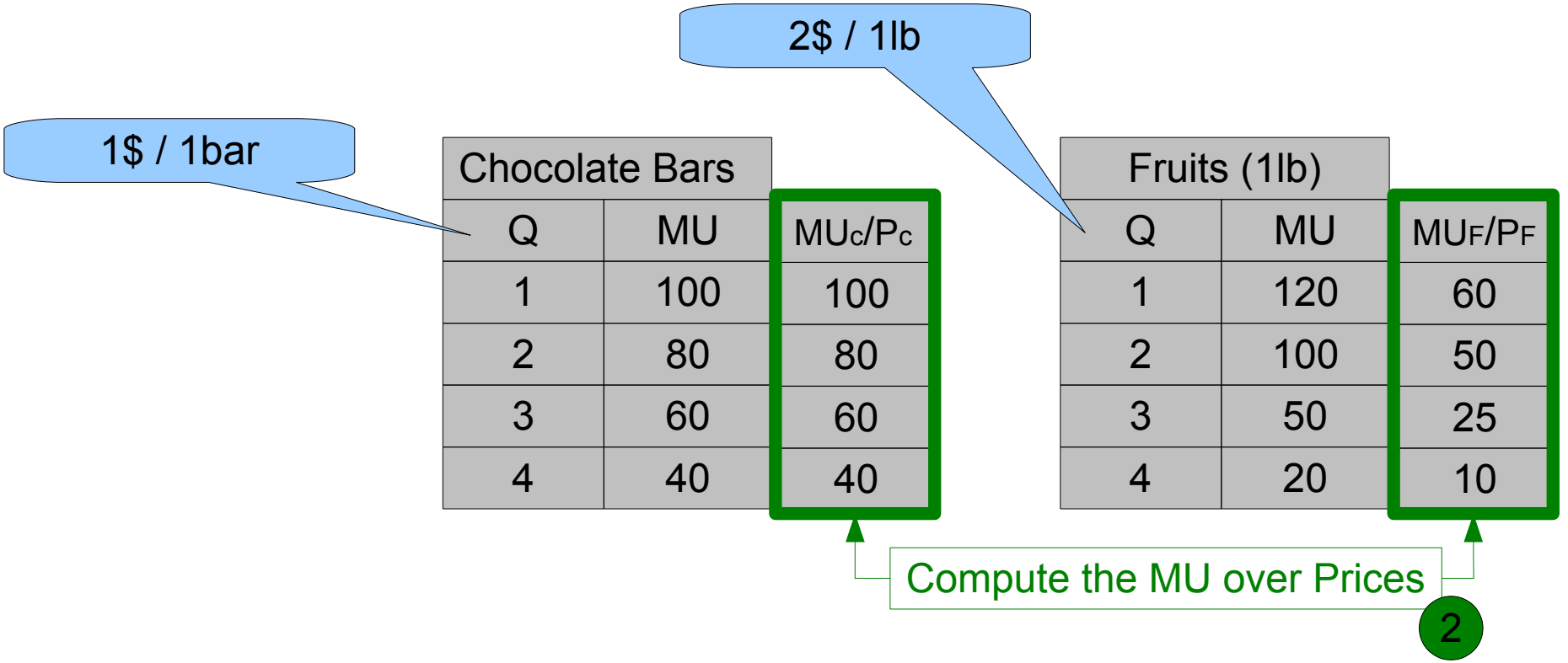
Q: How do we allocate our money given the marginal utilities ?
Fruits over chocolate bars ?

Diminishing utility (benefit) as we are getting more incremental units

Start by assigning the prices, assuming that the prices are Market-given: the consumer cannot change the price of a good

Utility Theory

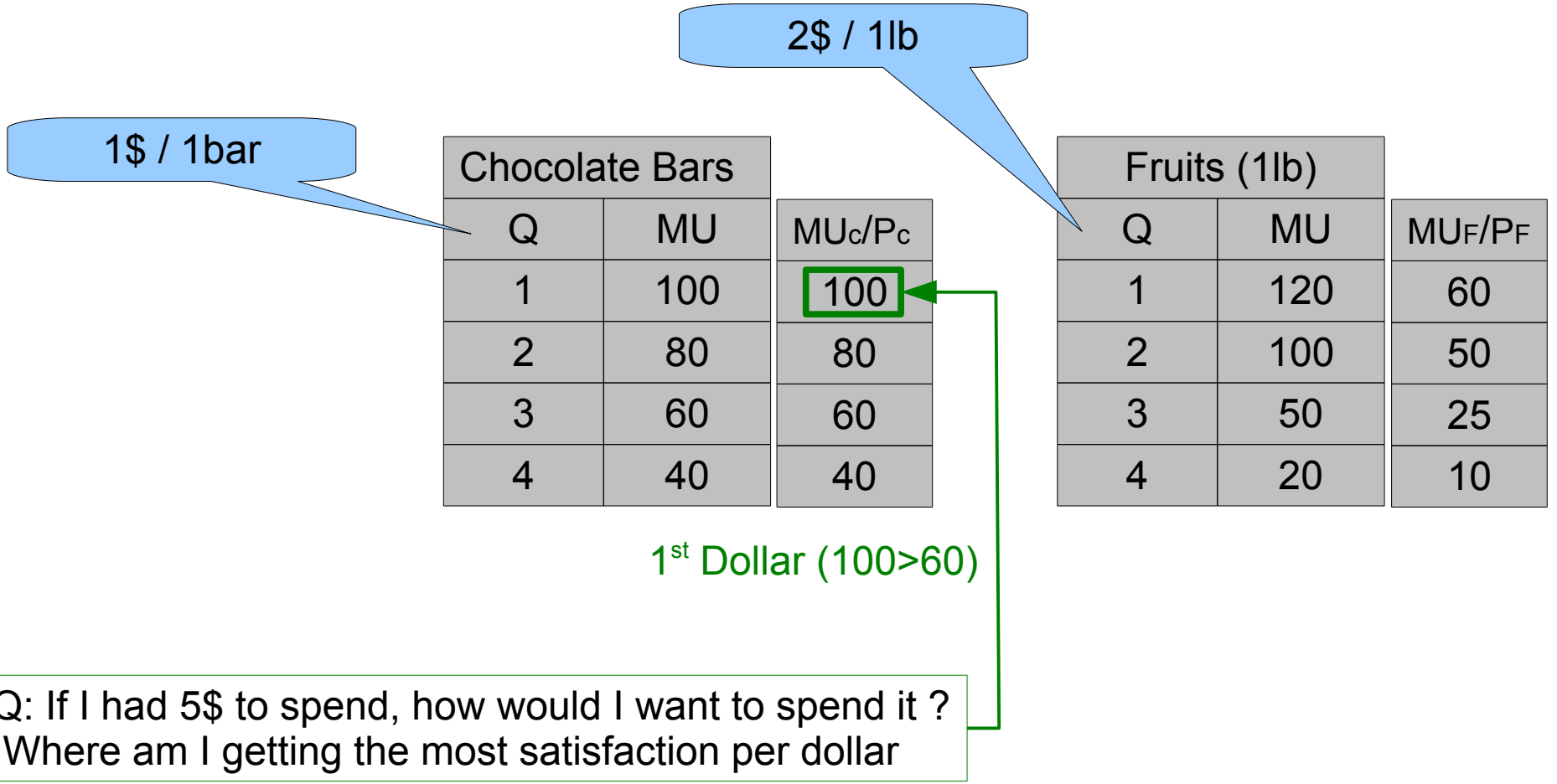
- Utility-Maximizing Rule



Q: If I had 5$ to spend, how would I want to spend it ?
Where am I getting the most satisfaction per dollar

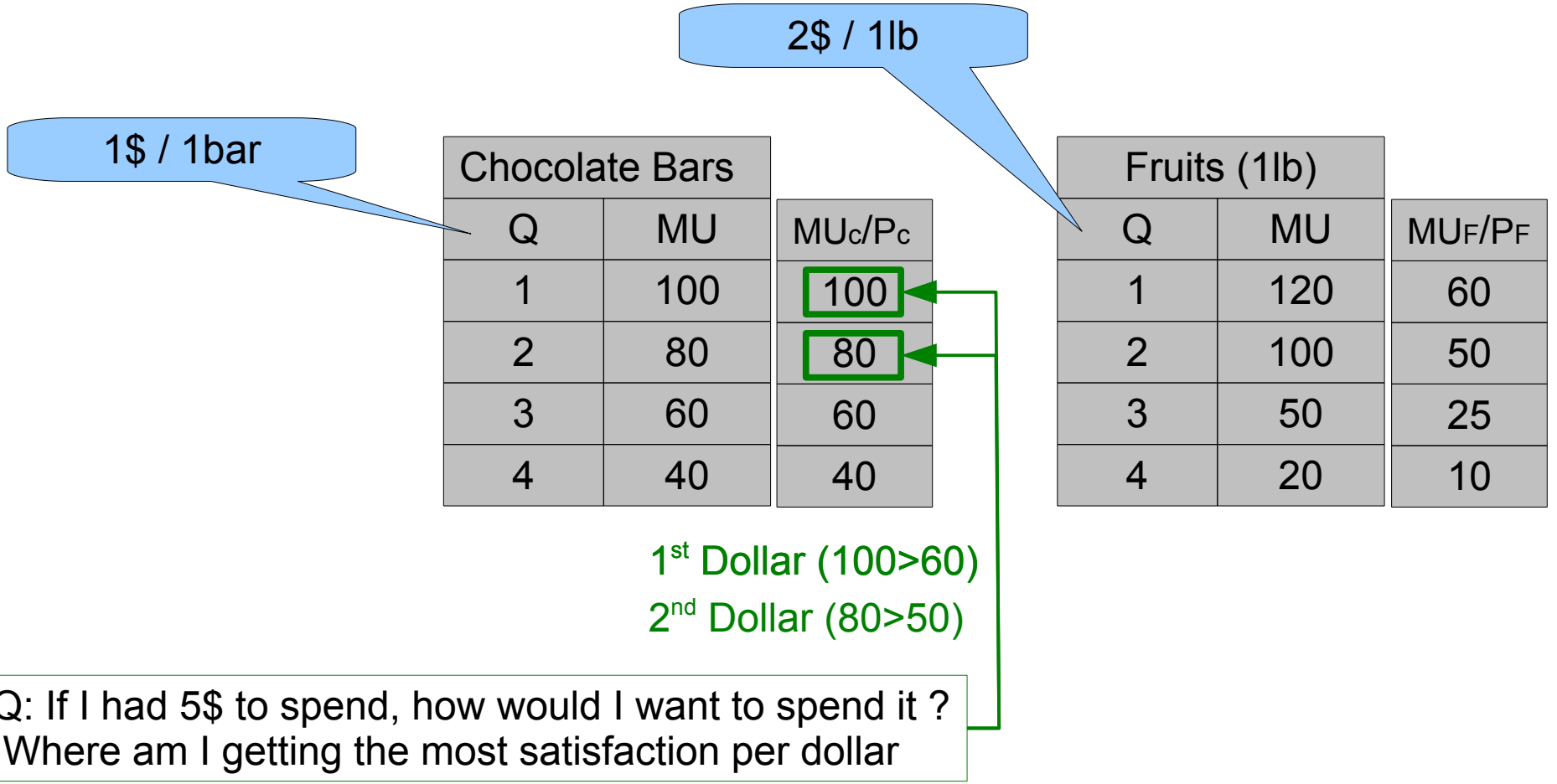
Utility Theory

- Utility-Maximizing Rule



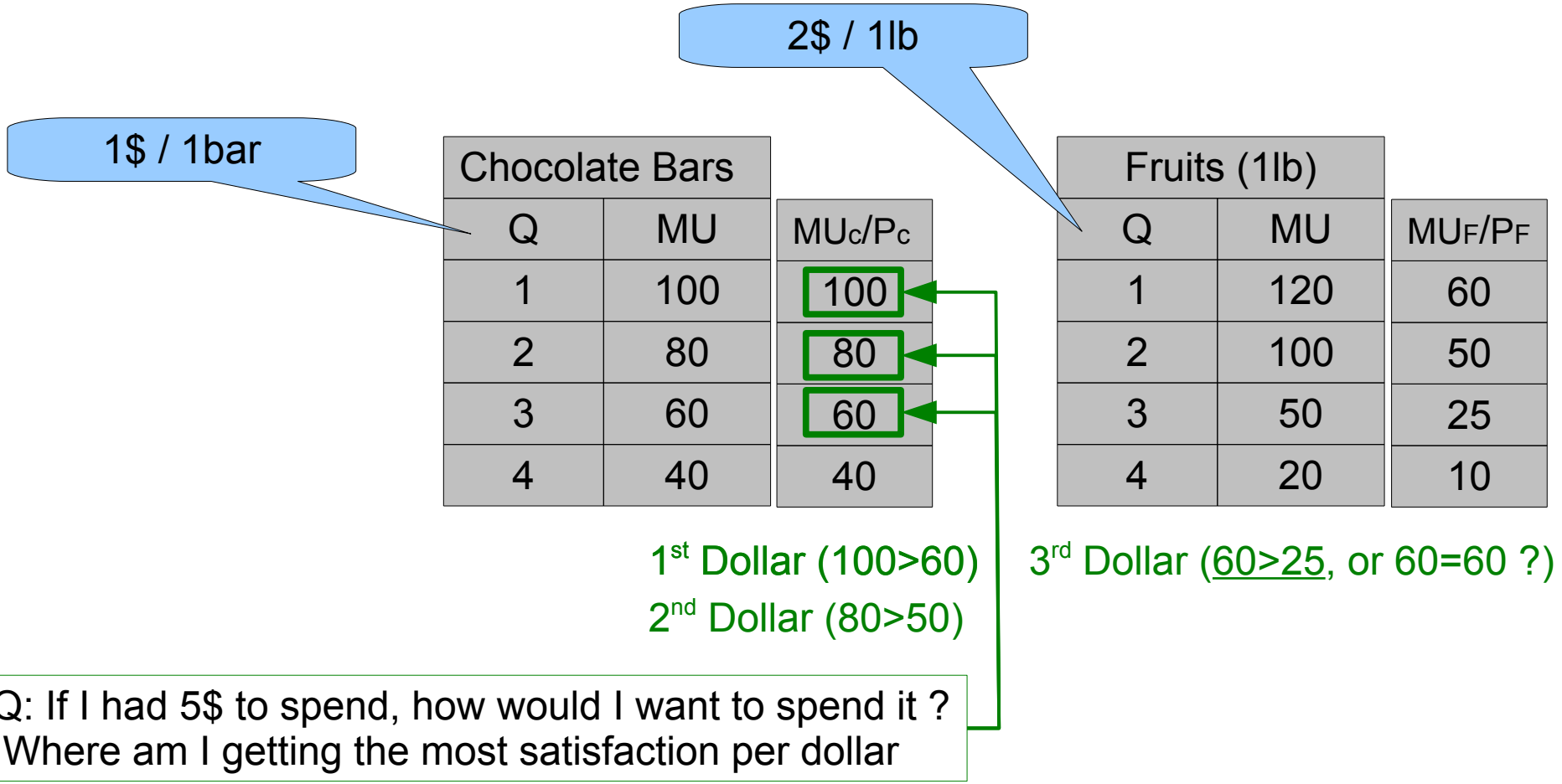
Utility Theory

- Utility-Maximizing Rule



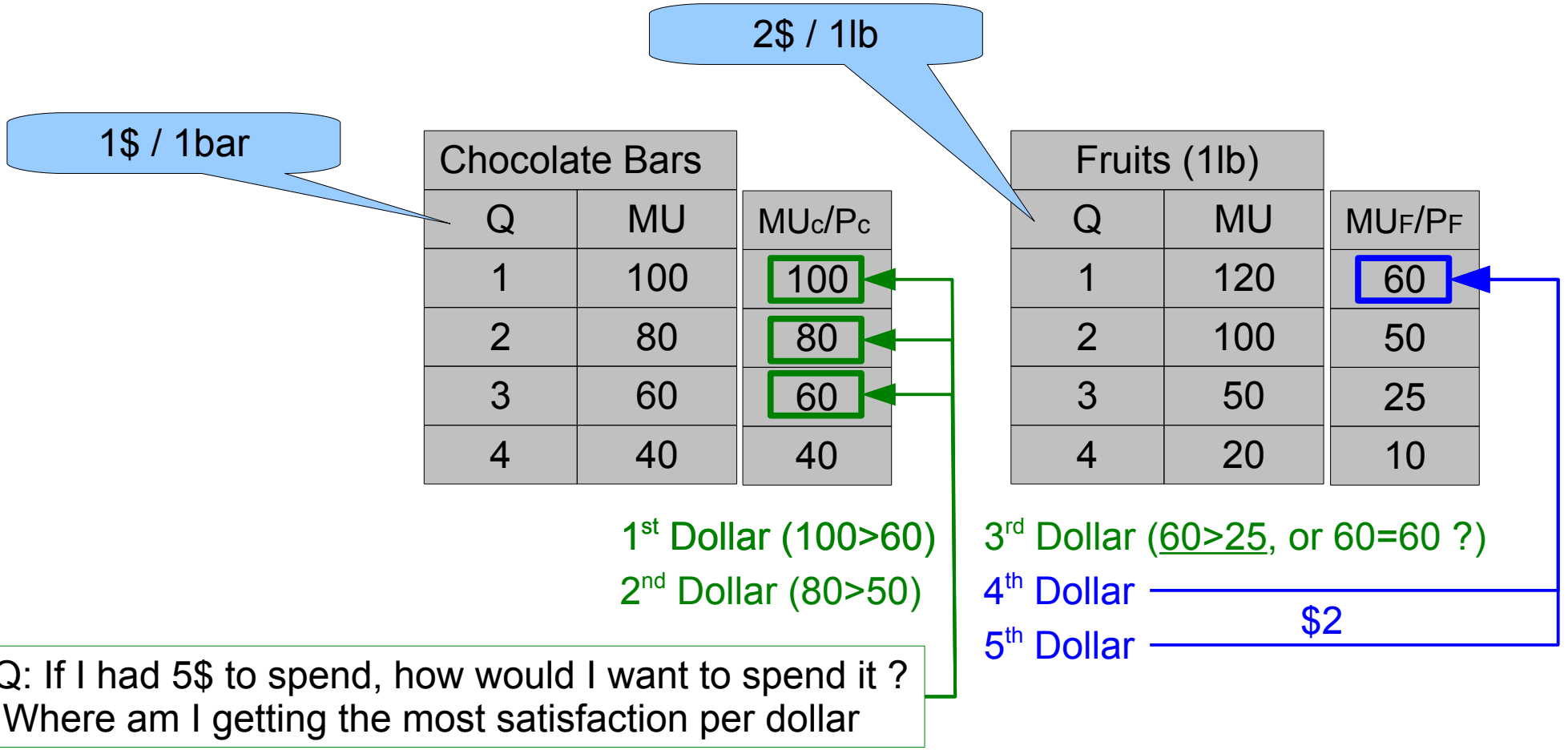
Utility Theory

- Utility-Maximizing Rule



Utility Theory

- Utility-Maximizing Rule



Utility Theory

- Utility-Maximizing Rule
- Based on the perceived marginal utilities for the first 3 dollars, the values of a chocolate bar is valued more then the marginal utility of one pound of fruits
- That is because one dollar's worth of chocolate bar would give more utility that one dollar's worth of fruits:

$$\frac{MU_C}{P_C} > \frac{MU_F}{P_F}$$

- Consumer Equilibrium

Utility Theory

- There is a need to automatize the process of decision making and preference elicitation :
  - ➔ We need a more formal and axiomatic description of utility
- Utility function
  - Input: the domain of the outcomes, going from physical goods (Example of the fruits, chocolate bars) to monetary prospect, actions, …
  - Output: scalar values, usually real numbers in the interval [0,1]
- The Game Theoretical axiomatization of utility function to describe the happiness and preferences of an agent

Utility Theory

- A game theoretical foundation as an attempt to match reality in a precise and coherent way
  - “The true source of uncertainty lies in the intentions of others”
- Invented by John von Neumann and Oskar Morgenstern
  - “Theory of Games and Economic Behavior” (1944)
- Rigidly (axiomatic!) mathematical with an emphasis on numerical quantities

Oskar Morgenstern and John von Neumann at Spring Lake, ca. 1946.

Utility Theory

- An axiomatic definition
- O, a finite set of outcomes. For $o_1, o_2 \in O$,
  - $o_1 \succcurlyeq o_2$: agent prefers o_1 to o_2
  - $o_1 \sim o_2$: agent is indifferent between o_1 and o_2
  - $o_1 \succ o_2$: agent strictly prefers between o_1 and o_2
- Preferences & uncertainty about the outcomes : lotteries
- A lottery is a probability distribution $[p_1 : o_1, \ldots, p_k : o_k]$ where $o_i \in O$, $p_i \in [0,1]$ and $\sum_{i=1}^{k} p_i = 1$

The only needed relation

  - ➔ Possibility to extend $\succcurlyeq$ to lotteries (lotteries seen as outcomes)

Utility Theory

i. *Completeness*: $\forall\ o_1, o_2 \in O,\ o_1 \succ o_2$ or $o_2 \succ o_1$ or $o_1 \sim o_2$

ii. *Transitivity*: if $o_1 \succcurlyeq o_2$ and $o_2 \succcurlyeq o_3$, then $o_1 \succcurlyeq o_3$ (proof: irrationality of the non-transitivity. E.g., *money pomp*)

iii. *Substitutability*: if $o_1 \sim o_2$ then $\forall$ sequences of one or more outcomes $o_3,\ldots, o_k$ and sets of probabilities $p_3,\ldots, p_k$ for which $p + \sum_{i=3}^{k} p_i = 1$,

$[p{:}o_1, p_3{:}o_3, \ldots, p_k{:}o_k] \sim [p{:}o_2, p_3{:}o_3, \ldots, o_k{:}o_k]$

iv. *Decomposability*: if $\forall o_i \in O,\ P_{l_1}(o_i) = P_{l_2}(o_i)$ then $l_1 = l_2$

- An agent is indifferent between lotteries that induce the same probabilities over outcomes (single lottery or nested lotteries)

v. *Monotonicity*: if $o_1 \succ o_2$ and $p > q$, then $[p{:}o_1, 1\text{-}p{:}o_2]$ $[q{:}o_1, 1\text{-}q{:}o_2]$

- The agents prefers the lottery that assigns the larger probability to o_1

vi. *Continuity*: if $o_1 \succcurlyeq o_2$ and $o_2 \succcurlyeq o_3$, then $\exists p$ [0,1] such that $o_2 \sim [p{:}o_1, 1\text{-}p{:}o_3]$

Utility Theory

- Based on the previous axioms : There exist a single-dimensional utility functions whose expected values agents want to maximize

  Theorem (*von Neumann-Morgenstern, 1944*)

  If a preference relation $\succ$ satisfies the axioms of *completeness*, *transitivity*, *substitutability*, *decomposability*, *monotonicity* and *continuity*, then there exists a function u:O→[0,1] satisfying :

$$u(o_1) > u(o_2) \text{ iff } o_1 \succ o_2 \qquad (1)$$

$$u([p_1:o_1, \ldots, p_k:o_k]) = \sum_{i=1}^{k} p_i \cdot u(o_i) \qquad (2)$$

- We start by treating certain aspects of the single dimensional case, before moving to the multidimensional case or Multi-attribute utility

Utility Theory

- Advantages of UT:
  - Possibility to embed the decision maker attitudes towards risk
  - Involve situations subject to uncertainty

Utility Theory

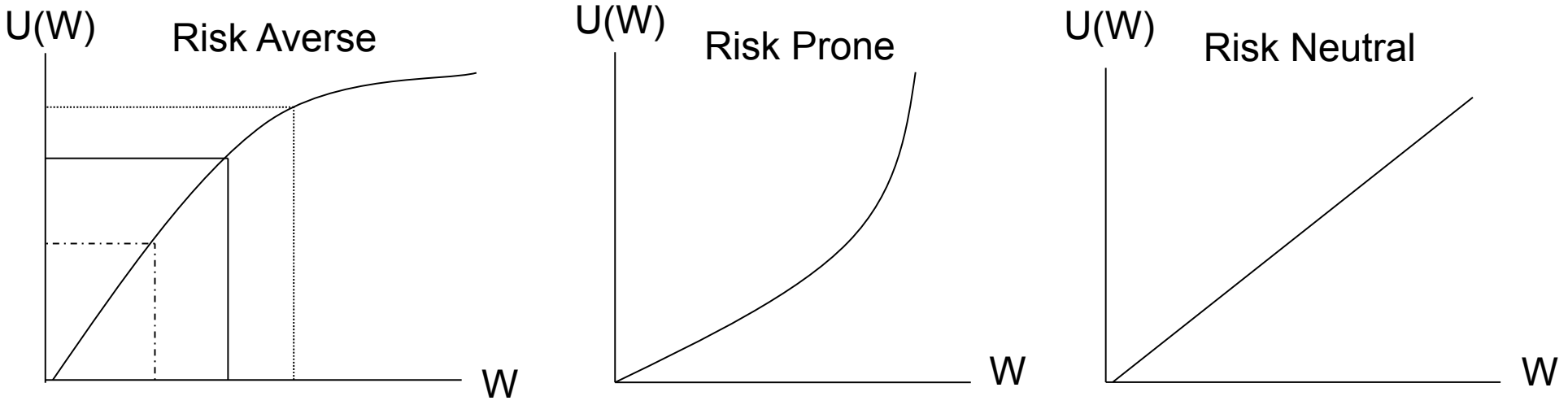
- Attitude towards risk
- Firstly, we define certain notions related to the context of risk attitude and uncertainty
  - Gamble: Action with more than one possible outcome, where each outcome has a probability of occurring. For instance, if the outcomes are good (G) and bad (B), denote the associated probabilities by pG and pB
  - Payoff: to each outcome we can assign a payoff, that could be expressed in terms of money: $cG and $cB
  - Utility of a payoff: u(cG) is the utility in the good situation while u(cB) is the utility in the bad situation (u'(c)=du/dc >0)
  - Expected Return: The expected return from the gamble is: ER=pG×cG+pB×cB
  - Expected Utility: from the gamble is: EU=pG×u(cG)+pB×u(cB)

Utility Theory

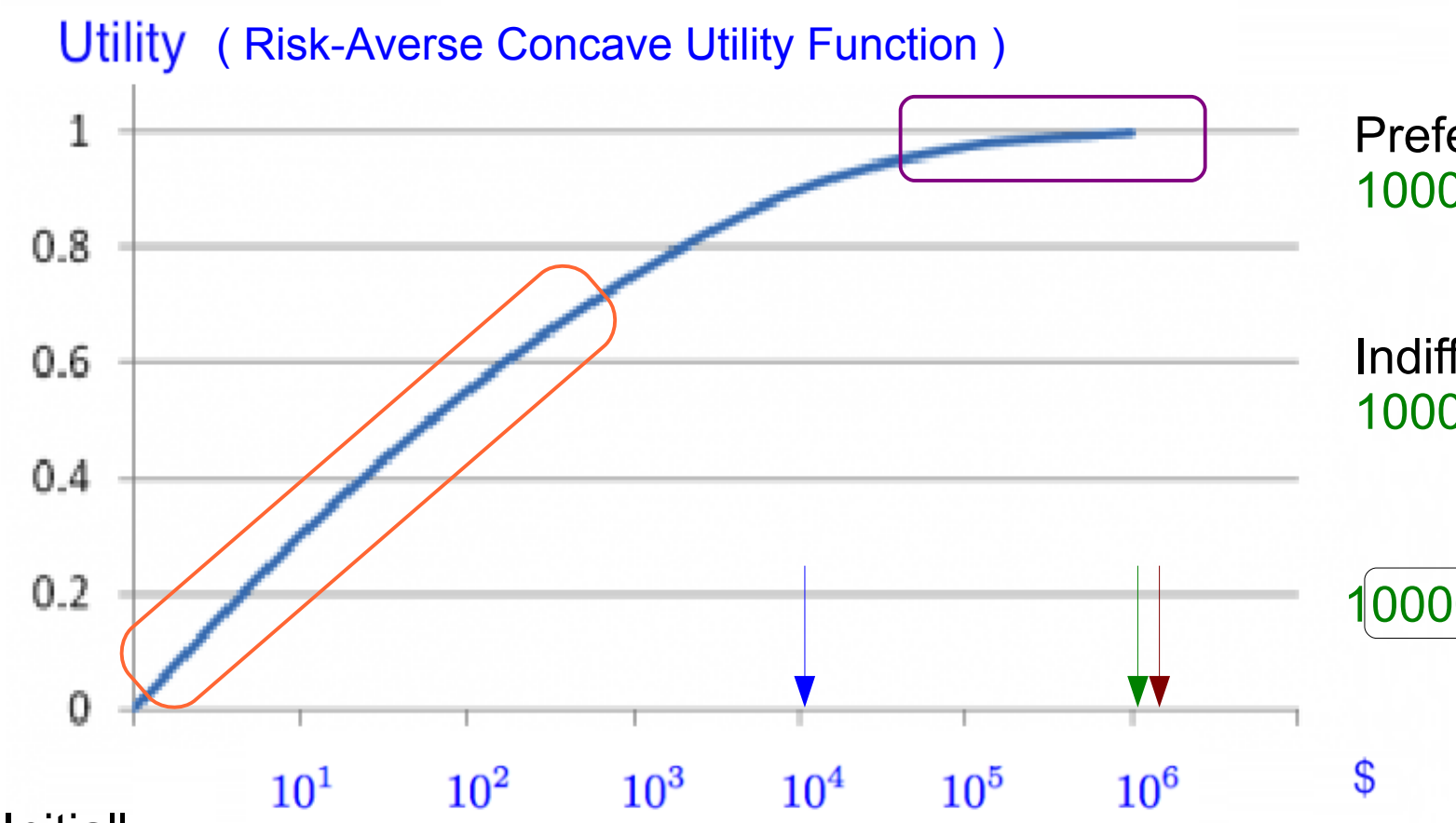
- Expected Utility rule
  - Assuming that we have a wealth of $W, and we are faced with a gamble. We have to decide whether or not to accept the gamble
  - If we accept:
    - The expected Utility is : $EU=pG\times u(cG)+pB\times u(cB)$
  - If we reject:
    - The certain utility is u(W)
- To decide, we can compare EU and u(W):
    - Accept if EU>u(W)
    - Reject if EU<u(W)
    - Indifferent if EU=u(W)

Utility Theory

- Attitude towards risk:
  - whether someone accepts a gamble or not, depends on his attitude to risk
- Three attitudes are defined
  1. The Risk Averse Person
  2. The Risk Neutral Person
  3. The Risk Prone Person



Utility Theory



Preference:
1000000$ > 10000$

Indifference:
1000000$ ~ 1000001$

1000000$ >>>> 1$

Initially:

- The utility function increases quickly and then flattens out
- Money is more valuable than additional sums of money once we are already rich

It represents a person who prefers not to take risks (risk-averse)

Utility Theory

- Several models for utility representation, elicitation and usage:
  - Preferences elicitation and risk analysis in decision making, additive utilities: “Decisions with Multiple Objectives: Preferences and Value Tradeoffs”, Ralph L. Keeney, Howard Raiffa
  - Utility distribution, with a structure similar to probabilities: “A Symmetric View of Utilities and Probabilities”, Yoav Shoham
  - Model inspired from Bayesian and Markov models: “Directional Decomposition of Multiattribute Utility Functions”, Ronen I. Brafman and Yagil Engel
  - Representation as random variables: “Utilities as random variables: Density estimation and structure discovery”, U. Chajewska and D. Koller
  - Utility assignment, based on the ME,when only partial information is available about the decision maker preferences: “Maximum Entropy Utility”, Ali E. Abbas
  - Graphical representation and a qualitative method, under *Ceteris Paribus* hypotheses: C. Boutilier, R. I. Brafman, C. Domshlak, H. H. Hoos and D. Poole (2004) "CP-nets: A Tool for Representing and Reasoning withConditional Ceteris Paribus Preference Statements"

Utility Theory

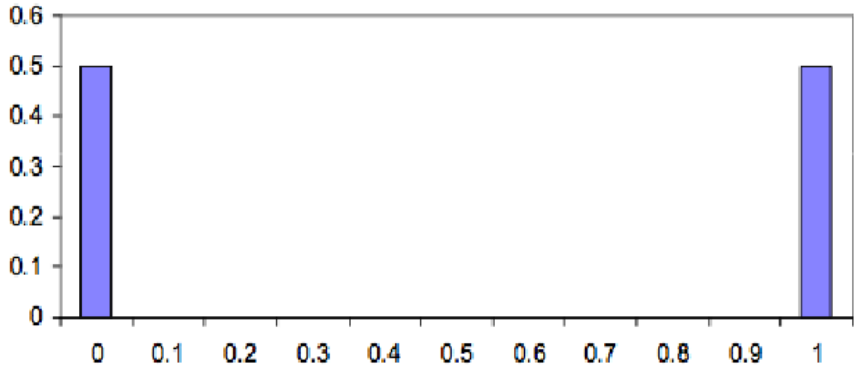
- Preferences Elicitation: understanding and obtaining a good (informative) and useful (convenient to reason with) description of the agent's objective
- Use the natural statements that the user is making, as well as the answers to questions the user finds intuitive
- Main categories of approaches:
  - Classical: real valued function on the space of possible outcomes (quantitative). But difficult to elicit from users
  - MAUT: describe a utility for each outcome, independently. Example: utility of different arrival times, utility of safety of each means of transportation, etc. Compare the outcomes by comparing utilities component-wise (dominance relations: Pareto Optimality)
  - Qualitative: People are more comfortable making qualitative preference statements
    - “I like this more than that”

Utility Theory

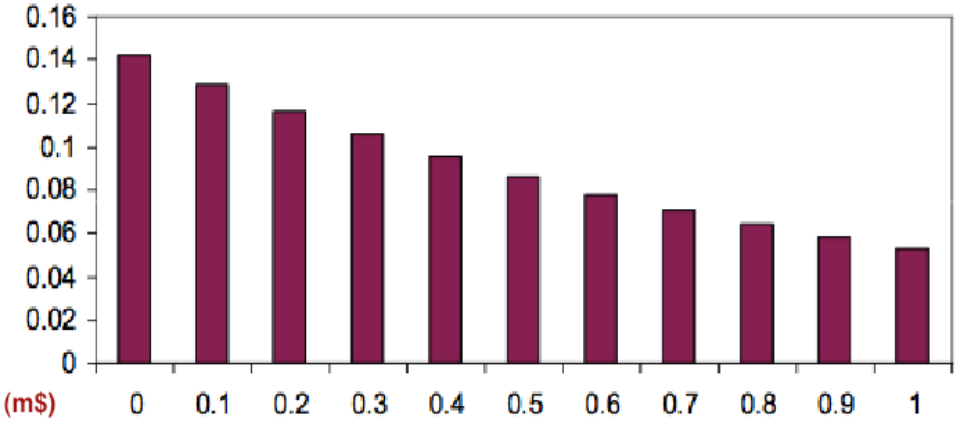
- Examples of qualitative methods
  - Doyle and Wellman (1991): Logic of relative desire
    - Reason about statements of the form "α is preferred to β"
    - Adopts Ceteris Paribus ("all else being equal") semantics
    - Extends the "logic of preference" of von Wright (mainly propositional)
  - Boutilier (1994)
    - Reason about statements of the form "*If p then it is better to have q*"
    - Any p-world with q is better than any p-world without q
    - Reasoning based on propositional non-monotonic logic

Utility Theory

- Example of Elicitation based on the ME principle
- Inferences on the basis of partial preference information ?
  - Utility assignment : Use the **utility function** whose utility density has ME subject to whatever preferences are known



Deal 1



Deal 2

Utility Theory

- The party problem

	Prospects	Dollar value ($)	Utility value
1	Outdoors, Sunny	100	1
2	Porch, Sunny	90	0.95
3	Indoors, Rainy	50	0.67
4	Indoors, Sunny	40	0.57
5	Porch, Rainy	20	0.32
6	Outdoors, Rainy	0	0

Utility Theory

- The party problem

	Prospects	Dollar value ($)	Utility value
1	Outdoors, Sunny	100	1
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6	Outdoors, Rainy	0	0

$$u_{maxent}(x) = \underset{u(x)}{argmax}\left(-\int_0^{100} u(x)\ln u(x)\,dx\right) \quad \text{s.t.} \quad \int_0^{20} u(x)\,dx = 0.32,\ \int_0^{40} u(x)\,dx = 0.57$$

$$\int_0^{50} u(x)\,dx = 0.67,\ \int_0^{90} u(x)\,dx = 0.95$$

$$\int_0^{100} u(x)\,dx = 1$$

Utility Theory

- Use of indicator functions over the intervals (moment constraints)

$$\int_0^{20} u(x)\,dx = 0.32 \quad \Rightarrow \quad \int_0^{20} u(x)\,dx = \int_0^{100} I_{20}(x)\,u(x)\,dx$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

- The ME utility is given by solving a *Lagrange* multipliers optimization problem

$$u_{maxent}(x) = e^{-\alpha_0 - 1 - \alpha_1 I_{20}(x) - \alpha_2 I_{40}(x) - \alpha_3 I_{50}(x) - \alpha_4 I_{90}(x)}, \; 0 \le x \le 100$$

Utility Theory

- Limits, Open problems:
  - The entire risk profile cannot be captured with a single number (EU)
  - "Utile" has no meaning to most people
  - No natural utility function: when should we use log or square root?
  - People violate axioms as it has been shown in Prospect Theory (Daniel Kahneman and Amos Tversky)

Thank you